3^{RD} SPECIAL MATHS CONTEST

2024 Edition: Second/Final Round

Solutions and Marking Scheme

Instructions for Marking

- *Every Problem is worth 7 points.*
- *Full solutions regardless of method is worth 7 points. 1 point may be deducted for fundamental error and 2 points for a more serious error which can be rectified without much trouble.*
- *Random attempts not in line with scheme are worth at most 1 point unless it can be shown to lead to a potential solution.*
- *Any substantial or insightful partial solution outside the scheme will be discussed by team in more details.*
- *Alternative approach under each point breakdown will receive appropriate points corresponding to progress made.*

1 Combinatorics

1.1 Problem

Problem 1: A company has $n+1$ men and n women. In order to form a committee that consists of more men than women from this company, how many ways can this be done? Give your answer in a closed form.

Key Idea: Can be expressed as summation involving combinations. There is a hidden symmetry to be exploited.

1.2 Solutions

Solution 1:
Let x denote the number of ways of forming a committee that consists of more men than

women from this company. Let y denote the number of ways of forming a committee that consists of less or equal number of men than women from this company (including the empty set).

Then $x + y$ is the total number of possible committee. So $x + y = 2^{2n+1}$ since there are $2n + 1$ people in total.

Next we show $x = y$. Consider the compliment committee made by remaining members. A committee is counted as part of x if and only if its compliment is counted as part of y. So we get a bijection and conclude that $x = y$.

Thus, $2x = 2^{2n+1} \implies x = 2^{2n} = 4^n$.

Solution 2:
Let x, y be same as before. Then

$$x = \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{k=0}^{i-1} \binom{n}{k}, y = \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{k=i}^{n} \binom{n}{k}.$$

Indeed, $x + y = \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{k=0}^{n} \binom{n}{k} = \sum_{i=0}^{n+1} \binom{n+1}{i} 2^n = 2^{n+1} \cdot 2^n = 2^{2n+1}$.
Now,

$$\begin{aligned} y &= \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{k=i}^{n} \binom{n}{k} \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{l=0}^{n-i} \binom{n}{n-l} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{n+1-j} \sum_{l=0}^{j-1} \binom{n}{n-l} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} \sum_{l=0}^{j-1} \binom{n}{l} \\ &= x. \end{aligned}$$

The rest follows as in solution 1.

Solution 3:
Let x be same as before. Then

$$x = \sum_{i=0}^{n+1} \binom{n+1}{i} \sum_{k=0}^{i-1} \binom{n}{k}$$

$$= \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i-1} \binom{n}{k} + \sum_{i=1}^{n+1} \binom{n}{i-1} \sum_{k=0}^{i-1} \binom{n}{k}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{l=n+1-i}^{n} \binom{n}{n-l} + \sum_{j=0}^{n} \binom{n}{n-j} \sum_{k=0}^{n-j} \binom{n}{k} \\
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{l=n+1-i}^{n} \binom{n}{l} + \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n}{k} \\
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{k=n+1-i}^{n} \binom{n}{k} + \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{n-i} \binom{n}{k} \\
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{n} \binom{n}{k} \\
&= 2^n \cdot 2^n = 4^n.
\end{aligned}
$$

1.3 Marking scheme

Type 1: Based on solution 1

- [**1 Point**] Consider y (well described).
- [**2 Points**] Deducing $x + y = 2^{2n+1}$.
- [**3 Points**] Showing $x = y$. This may include considering the compliment committee (and the associated number x') and use of bijection twice to get $x = x' = y$.
- [**1 Point**] Finalizing to get $x = 4^n$.

Type 2: Based on solutions 2 and 3

- [**2 Points**] Translating x into the right summation.
- [**5 Points**] Simplifying the sum:
  - [**1 Point**] Splitting the sum or taking compliment.
  - [**1 Point**] Substituting the summation i appropriately.
  - [**1 Point**] Substituting the summation k appropriately.
  - [**1 Point**] Applying $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
  - [**1 Point**] Using the symmetry hidden in $\binom{n}{k} = \binom{n}{n-k}$.

2 Geometry

2.1 Problem

Problem 2: Let $ABCD$ be a convex quadrilateral with $B\hat{C}D = 90°$ and $B\hat{A}D = B\hat{D}A$. Suppose E lies on line segment AC such that $D\hat{A}E = C\hat{D}E$ and $|BC| = |AE|$.
Find the measure of $A\hat{C}B$.

Key Idea: Application of various basic theorems involving length.

2.2 Solutions

Solution 1:
By scaling, let $|CE| = 1, |CD| = t$. Then $D\hat{A}E = C\hat{D}E \implies \triangle CAD \sim \triangle CDE$. Thus, $\frac{|AC|}{|CD|} = \frac{|DC|}{|CE|} \implies |AC| = t^2 \implies |AE| = t^2 - 1 > 0$.

Next, $|BC| = |AE| \implies |BC| = t^2 - 1$. So, $B\hat{C}D = 90° \implies |BD|^2 = |BC|^2 + |CD|^2 = (t^2-1)^2 + t^2 = t^4 - t^2 + 1$.
Now, from $B\hat{A}D = B\hat{D}A$, have $|AB| = |BD|$, so $|AB|^2 = t^4 - t^2 + 1$ also.
Finally, $\cos(A\hat{C}B) = \frac{|AC|^2+|BC|^2-|AB|^2}{2|AC|\cdot|BC|} = \frac{t^4+(t^2-1)^2-(t^4-t^2+1)}{2t^2\cdot(t^2-1)} = \dfrac{1}{2}$.
We conclude that $A\hat{C}B = 60°$.

Solution 2:
Let F be on opposite side of BC to A such that $\triangle ABC \equiv \triangle DBF$ with same orientation.
Let G be on segment DF such that $|DG| = |EC|, |GF| = |AE| = |BC| = |BF|$.

Then $D\hat{A}E = C\hat{D}E \implies \triangle CAD \sim \triangle CDE$. Thus, $\frac{|AC|}{|CD|} = \frac{|DC|}{|CE|} \implies \frac{|FD|}{|DC|} = \frac{|CD|}{|DG|} \implies \triangle DFC \sim \triangle DCG \implies C\hat{F}G = D\hat{C}G$.

So, CD is tangent to the circumcircle of $\triangle CFG$ whose centre then lies on BC and the perpendicular bisector of CF. We conclude that B is that centre.

Consequently, $|BG| = |BF| = |GF|$ so that $A\hat{C}B = D\hat{F}B = 60°$.

2.3 Marking scheme

Type 1: Based on solution 1. (Deducing lengths).

- **[2 Points]** $|AC|$.
- **[1 Point]** $|BC|$.
- **[1 Point]** $|BD|$.
- **[1 Point]** $|AB|$.

- **[2 Points]** Using cosine rule to get final answer.

Type 2: Based on solution 2. (construction)

- **[1 Point]** Construction of F.
- **[1 Point]** Construction of G.
- **[1 Point]** Deducing $\triangle CAD \sim \triangle CDE$.
- **[2 Points]** Deducing $C\hat{F}G = D\hat{C}G$.
- **[1 Point]** Deducing B is center of $\triangle CFG$.
- **[1 Point]** Deducing $\triangle BFG$ is equilateral.

3 Algebra

3.1 Problem

Problem 3: Consider three sequences of real numbers $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ satisfying the following for $k \geq 0$:

$$x_{k+1} = (y_k + z_k)^2 - x_k^2,$$
$$y_{k+1} = (z_k + x_k)^2 - y_k^2,$$
$$z_{k+1} = (x_k + y_k)^2 - z_k^2.$$

Can any two of the sequences be periodic given that $x_0 \neq y_0, y_0 \neq z_0, z_0 \neq x_0$?

Key Idea: Exploit symmetry and consider growth rate as opposed to bounded-ness of periodic sequences.

3.2 Solutions

Solution 1:
Let $s_n = x_n + y_n + z_n$. Then $s_{k+1} = s_k^2, k \geq 0$. So, $s_n = s_0^{2^n}$. Suppose $s_0 = 0$, then have $s_k = 0 \implies x_{k+1} = y_{k+1} = z_{k+1} = 0$. This is a contradiction, therefore we conclude that $s_0 \neq 0$.

Now $x_{k+1} - y_{k+1} = (y_k + z_k)^2 - (z_k + x_k)^2 - x_k^2 + y_k^2 = (y_k - x_k)[x_k + y_k + 2z_k] + (y_k - x_k)[x_k + y_k] = 2s_k(y_k - x_k)$.

Thus, $\frac{x_{k+1} - y_{k+1}}{s_{k+1}} = (-2)[\frac{x_k - y_k}{s_k}]$. Inductively, have $\frac{x_n - y_n}{s_n} = (-2)^n[\frac{x_0 - y_0}{s_0}]$.

Finally, $x_n - y_n = (-2)^n s_0^{2^n - 1}(x_0 - y_0) \neq 0$. Similarly, $y_n - z_n = (-2)^n s_0^{2^n - 1}(y_0 - z_0) \neq 0$ and $z_n - x_n = (-2)^n s_0^{2^n - 1}(z_0 - x_0) \neq 0$.

Therefore suppose any two sequences are periodic, say x_n periodic with period $M \geq 1$ and y_n periodic with period $N \geq 1$. Then both x_n and y_n have period $T = MN \geq 1$. This implies $x_n - y_n$ is also periodic with period T.

So $(-2)^{kT} s_0^{2^{kT}-1}(x_0 - y_0) = x_0 - y_0, k \geq 1 \implies (-2)^T s_0^{2^T-1} = 1 \implies (-2)^{kT} s_0^{k(2^T-1)} = 1 \implies s_0^{k(2^T-1)} = s_0^{2^{kT}-1}$. In particular, $s_0^{2(2^T-1)} = s_0^{2^{2T}-1} \implies s_0^{(2^T-1)^2} = 1 \implies s_0 = 1$ or $2^T - 1 = 0$. So must have $s_0 = 1$. Next, recall $(-2)^{kT} s_0^{2^{kT}-1}(x_0 - y_0) = x_0 - y_0, k \geq 1$, so $(-2)^{kT} = 1, k \geq 1$. This is a contradiction.
Hence, at most one of the sequence is periodic.

On further note: Have $x_n = (-2)^n s_0^{2^n-1} x_0 + w_n, y_n = (-2)^n s_0^{2^n-1} y_0 + w_n, z_n = (-2)^n s_0^{2^n-1} z_0 + w_n \implies s_0^{2^n} = s_n = x_n + y_n + z_n = (-2)^n s_0^{2^n} + 3w_n \implies w_n = s_0^{2^n}[\frac{1-(-2)^n}{3}]$.

Thus,

$$x_n = (x_0 + y_0 + z_0)^{2^n-1}\left[x_0\left(\frac{1-(-2)^{n+1}}{3}\right) + (y_0 + z_0)\left(\frac{1-(-2)^n}{3}\right)\right].$$

Solution 2:
Suppose not. Let $s_n = x_n + y_n + z_n$ as before.
Similar to solution 1, may assume x_n and y_n are both periodic with period N. It follows that if $z_{k+mN} = z_{k+m'N}$ for some $k \geq 0, m > m' \geq 0$, then z_n is eventually periodic with period $(m - m')N$ since the triple (x_n, y_n, z_n) is defined inductively.
Indeed, $x_{mN+1} = (y_{mN} + z_{mN})^2 - x_{mN}^2 \implies x_1 = (y_0 + z_{mN})^2 - x_0^2$. So there are two possible values for z_{mN}. Thus, two of z_0, z_N and z_{2N} are equal by pigeonhole principle. We conclude that z_n is eventually periodic with period $2N$. Consequently, $s_n = s_0^{2^n}$ is also eventually periodic. So, $s_0 = \pm 1$ and $s_n = 1, n \geq 1$.

Thus, $x_{k+1} = 1 - 2x_k, y_{k+1} = 1 - 2y_k, z_{k+1} = 1 - 2z_k, k \geq 1$. Have $3x_{k+1} - 1 = -2(3x_k - 1) \implies 3x_n - 1 = (-2)^{n-1}(3x_1 - 1)$ is periodic. So $x_n = 1/3, n \geq 1$ and similarly $y_n = 1/3, n \geq 1$. This is a contradiction.

3.3 Marking scheme

- **[2 Points]** Introducing and deducing a closed form for s_n.
- **[1 Point]** Show $s_0 \neq 0$.

Proof by contradiction [Assume x_n and y_n are periodic]:
Type 1: Based on x_n and y_n only.

- **[1 Point]** Deducing $\frac{x_n - y_n}{s_n}$ is exponential.

- **[1 Point]** Observing $x_n - y_n$ is periodic (and hence bounded).
- **[2 Points]** Completing the rest of the proof.

Type 2: Involving z_n.

- **[2 Points]** Deducing z_n is eventually periodic.
- **[1 Point]** Deducing $s_n = 1$ for large values of n.
- **[1 Point]** Finding x_n and concluding.

4 Number Theory

4.1 Problem

Problem 4: Let $(a_n)_{n\geq 0}$ be a sequence of rational numbers satisfying $a_{n+2}a_n^2 + a_{n+1}^3 = 4^n a_{n+1}a_n^4$ for $n \geq 0$. Suppose $a_0 = 1$ and $a_1 \neq 0$ is an integer.
Show that $(1+a_1^2)^{2^n} - 4^n a_n^2$ is the square of an integer for all $n \geq 0$.

Key Idea: Inductive application of Pythagorean triples.

4.2 Solutions

Solution 1:
Let $b_n := \frac{a_{n+1}}{a_n}$. Then $a_{n+2}a_n^2 + a_{n+1}^3 = 4^n a_{n+1}a_n^4 \implies \frac{a_{n+2}}{a_{n+1}} + [\frac{a_{n+1}}{a_n}]^2 = 4^n a_n^2 \implies b_{n+1} = 4^n a_n^2 - b_n^2, a_{n+1} = a_n b_n$. Since $a_0 = 1, b_0 = a_1$ are integers, it follows inductively that a_n and b_n are both integers.

Now let $c_n = 4^n a_n^2 + b_n^2$. Then, $c_{n+1} = 4^{n+1}a_{n+1}^2 + b_{n+1}^2 = 4^{n+1}a_n^2 b_n^2 + (4^n a_n^2 - b_n^2)^2 = (4^n a_n^2 + b_n^2)^2 = c_n^2$. Inductively, have $c_n = c_0^{2^n} = (a_0^2 + b_0^2)^{2^n} = (1+a_1^2)^{2^n}$.

Hence, $(1+a_1^2)^{2^n} - 4^n a_n^2 = b_n^2$ is the square of an integer for all $n \geq 0$.

Solution 2:
Observe that $\frac{a_{n+2}^2}{a_{n+1}^2} = \left[4^n a_n^2 - \frac{a_{n+1}^2}{a_n^2}\right]^2 = \left[4^n a_n^2 + \frac{a_{n+1}^2}{a_n^2}\right]^2 - 4^{n+1}a_{n+1}^2$, so $4^{n+1}a_{n+1}^2 + \frac{a_{n+2}^2}{a_{n+1}^2} = \left[4^n a_n^2 + \frac{a_{n+1}^2}{a_n^2}\right]^2$.
Thus, inductively, have $4^n a_n^2 + \frac{a_{n+1}^2}{a_n^2} = \left[a_0^2 + \frac{a_1^2}{a_0^2}\right]^{2^n} = [a_1^2+1]^{2^n}$.

Finally, if a_n, a_{n+1} are integers with $a_n | a_{n+1}$, then from $\frac{a_{n+2}}{a_{n+1}} = 4^n a_n^2 - \frac{a_{n+1}^2}{a_n^2}$, have a_{n+1}, a_{n+2} are integers with $a_{n+1}|a_{n+2}$. So inductively, $[a_1^2+1]^{2^n} - 4^n a_n^2 = \frac{a_{n+1}^2}{a_n^2}$ is a perfect square since a_0, a_1 are integers with $a_0 | a_1$.

4.3 Marking scheme

- [**3 Points**] Show a_n is always an integer with $a_n | a_{n+1}$. This needs to be shown simultaneously in case of induction.
- [**2 Points**] Manipulating the recurrence equation in a way that leads to something inductive.
- [**1 Point**] Properly applying induction.
- [**1 Point**] Deducing the expression in question is equal to the square of $\frac{a_{n+1}}{a_n}$.