3^{RD} SPECIAL MATHS CONTEST

2024 Edition: Second/Final Round

Instructions

- *Duration: 4 hours*
- *Use white plane A4 papers for your work and write your name on each paper you submit.*
- *Ask questions about any problem you do not understand within the first 30 minutes of the exam.*
- *All Theorems and Lemmas must be clearly stated together with their names or with proof.*
- *Answer each problem with proof.*
- *All problems are worth equal points.*

Problem 1: A company has $n+1$ men and n women. In order to form a committee that consists of more men than women from this company, how many ways can this be done? Give your answer in a closed form.

Problem 2: Let $ABCD$ be a convex quadrilateral with $B\hat{C}D = 90^\circ$ and $B\hat{A}D = B\hat{D}A$. Suppose E lies on line segment AC such that $D\hat{A}E = C\hat{D}E$ and $|BC| = |AE|$.
Find the measure of $A\hat{C}B$.

Problem 3: Consider three sequences of real numbers $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ satisfying the following for $k \geq 0$:

$$x_{k+1} = (y_k + z_k)^2 - x_k^2,$$
$$y_{k+1} = (z_k + x_k)^2 - y_k^2,$$
$$z_{k+1} = (x_k + y_k)^2 - z_k^2.$$

Can any two of the sequences be periodic given that $x_0 \neq y_0, y_0 \neq z_0, z_0 \neq x_0$?

Problem 4: Let $(a_n)_{n\geq 0}$ be a sequence of rational numbers satisfying $a_{n+2}a_n^2 + a_{n+1}^3 = 4^n a_{n+1} a_n^4$ for $n \geq 0$. Suppose $a_0 = 1$ and $a_1 \neq 0$ is an integer.
Show that $(1 + a_1^2)^{2^n} - 4^n a_n^2$ is the square of an integer for all $n \geq 0$.

Happy Solving!