Special Maths Contest 2024

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1st Round Problems

November 23, 2024

1. Given $x^2 + y^2 + z^2 = 6$ and $xy + yz + zx = -1$, what is the value of $(x-y)(y-z) + (y-z)(z-x) + (z-x)(x-y)$?
A) -1
B) 0
C) 4
D) 7
E) -7

2. x, y, z are nonnegative integers satisfying $3x + 5y + 7z = 103$. What is the minimum possible value of $x + y + z$?
A) 14
B) 15
C) 20
D) 21
E) 33

3. How many ordered triples (x_1, x_2, x_3) of positive odd integers satisfy $x_1 + x_2 + x_3 = 11$.
A) 3
B) 6
C) 15
D) 45
E) 78

4. Let $PQRS$ be a convex quadrilateral such that $PQ \parallel RS, |QR| = |PS|, \angle RPS = 90$. Then the following are possible except...
A) $|PQ| = |PS|$
B) $|PQ| = |RS|$
C) $2|PQ| = |RS|$
D) $2|PS| = |RS|$
E) $|PS| = |RS|$

5. Given that $(u+v+w-uvw)^2+(uv+vw+wu-1)^2$ can be factorized into n non-constant factors, what is the maximum possible value of n.
A) 0
B) 1
C) 2
D) 3
E) 4

6. What is the second digit from the right of 2^{2024}?
A) 1
B) 3
C) 5
D) 7
E) 9

7. Let $n = 2^2 3^3 5^5$. How many positive integer divisors of n are less than 600?
A) 30
B) 35
C) 36
D) 37
E) 72

8. ABC is an isosceles triangle with $\angle ABC = 120°$. $ABGH$ and $CBIJ$ are squares constructed on the exterior of the triangle ABC. What is the value of $\angle HJG + \angle HIG$?
A) 15°
B) 20°
C) 30°
D) 40°
E) Cannot be determined

9. $\overline{abc}_{10}$ is a number with distinct non-zero digits. The digits of this number are rearranged such that no digit maintains its original position. There are 2 possible permutations satisfying this. What is the maximum possible difference of the two numbers formed by this permutation?
A) 198
B) 783
C) 817
D) 912
E) 981

10. A polydivisible number in a base system, b, is a positive integer such that the first k digits of that number is divisible by k. For example, 220_3 (1 divides $2_3 = 2_{10}$, 2 divides $22_3 = 8_{10}$ and 3 divides $220_3 = 24$) is polydivisible, but 111_3 is not (1 divides $1_3 = 1_{10}$, 2 divides $11_3 = 4_{10}$ but 3 does not divide $111_3 = 13$). How many polydivisible numbers of length 6 are there in base 3? (The number should not start with 0)
A) 1
B) 2
C) 3
D) 4
E) No such number

11. On two opposite sides of a unit square, an equilateral triangle of same length is constructed in the interior of the square. What is the area of the region common to both triangles?
A) $\frac{2\sqrt{3}-3}{3}$
B) $\frac{2-\sqrt{3}}{2\sqrt{3}}$
C) 1/6
D) $\frac{3-\sqrt{3}}{6}$
E) The triangles do not intersect

12. Let $m, n \in Z$. Let $a = \gcd(m^5, n^5+1)$, $b = \gcd(m^5+1, n^5)$, $c = \gcd(m^{10} - n^5 - 1, n^{10} - m^5 - 1, (mn)^5)$ and $d = \gcd(m^{10} + m^5, n^{10} + n^5)$. Then have
A) $\min(a, b) = 1$
B) $c = 1$
C) $c > d$
D) $ab = c$
E) $ab = d$

13. Let $Q(x)$ be a polynomial in x such that for some real constants a, b, c, d, have $(x^2 + ax + b) \cdot Q(x) = x^5 + cx + d$, then $Q(a) =?$
A) $2ab - a^3$
B) ac/b
C) $d - ac$
D) $(ad - bc)/b^2$
E) ab

14. The letters of the word $'BRASS'$ are rearranged in such a way that no letter remains in the same position. How many ways can this be done?
A) 6
B) 12
C) 18
D) 24
E) 60

15. Let A, B, C, D be collinear points in the order given with segments $|AB| = 4, |BC| = 9, |CD| = 6$. Let E be a point with DE perpendicular to CD, and $|DE| = 9$. Let $a = \angle DAE, b = \angle DBE, c = \angle DCE$. Which of the following holds?
A) $a + b = c$
B) $a + c = 90$
C) $c = 2a$
D) $a + b = 60$
E) $a + b + c = 90$

16. Suppose $p \equiv 1(mod 12)$ and $1 \cdot 2 \cdot 3 \cdots (\frac{p-1}{3}) \equiv -2(mod p)$. Then $(\frac{p+2}{3}) \cdot (\frac{p+5}{3}) \cdots (\frac{2p-2}{3}) \equiv x(mod p)$, where...
A) 3
B) 70
C) 165
D) $\frac{p-4}{3}$
E) $\frac{p-1}{4}$

17. Let (x, y, z) denote the coordinates of a point lying on the sphere of unit radius centered at the origin $(0, 0, 0)$. Find the minimum possible value of $2x + 3y + 6z$.
A) 0
B) 7
C) -7
D) -6
E) $-\frac{11}{\sqrt{3}}$

18. How many ways can 2024 identical candies be shared among Pius, Williams and Mmesomachi, so that Pius always gets the most and Williams always gets the least, no one receives zero candy and no two receive the same number of candies?
A) $673 \cdot 674$
B) $1011 \cdot 2023$
C) $673 \cdot 1012$
D) $337 \cdot 1010$
E) $337 \cdot 1011$

19. $ABCD$ is a convex quadrilateral with $\overline{AB} = 24$, $\overline{BC} = 15$, $\overline{CD} = 7$, $\overline{DA} = 20$ and $\overline{BD} = 20$. What is the length of $\overline{AC}$?
A) 24
B) 21
C) $22\frac{1}{2}$
D) $23\frac{2}{5}$
E) Not enough information for a specific answer

20. Consider a sequence of **integers** $a_n, n \geq 0$, satisfying $a_{n+3} = 3a_{n+1} - 2a_n$, for all $n \geq 0$. Which of the following is such a sequence?
A) $\frac{(-2)^n+3n-1}{9}$
B) $(-2)^{n-1} - 3n/2 + 1/2$
C) $2^n - 1$
D) $n^2 + n + 1$
E) $2^{-n} + n - 1$

21. Let $x = \frac{a^2-bc}{a+b+c}, y = \frac{b^2-ca}{a+b+c}, z = \frac{c^2-ab}{a+b+c}$. Similarly, let $u = \frac{x^2-yz}{x+y+z}, v = \frac{y^2-zx}{x+y+z}, w = \frac{z^2-xy}{x+y+z}$. Have...
A) $au = x^2$
B) $a + u = 2x$
C) $u + v + w = x + y + z$
D) $u = a$
E) $u = x$

22. How many numbers in base 3, between 1_3 and 1000000_3 have each of the digits of the base 3 number system. $(0, 1, 2)$?
A) 488
B) 128
C) 537
D) 360
E) 540

23. Let Ω be a circle with triangle ABC lying in its interior. Extend AB to meet Ω at A_1 and B_2 beyond A and B respectively, extend BC to meet Ω at B_1 and C_2 beyond B and C respectively, extend CA to meet Ω at C_1 and A_2 beyond C and A respectively. Suppose $A_1A_2 \parallel BC, AA_2 = CA = 3m, BB_2 = CC_2 = 4m^2 - 1$, and $BB_1 = AB = x$. Then must have x equals...
A) 1
B) $2m^2 - 2$
C) $4m^2 + 1/2$
D) $2m$
E) $2m^2 + 1$

24. Let p and q be distinct prime numbers such that $(p-1, q-1)|12$ and $q|2^{p-1} - 1$. How many possible values does q have?
A) 3
B) 4
C) 5
D) At least 12
E) Infinitely many

25. We say a summation of two positive integers is *permutational* if the digits of the result of the sum has digits which is the permutation of one of the summands. For an example $45 + 9 = 54$ is a *permutational* sum becase 54 is a permutation of 45. How many *permutational* sums, with summands less than 1000 are there that involve 27 as one of the summands?
A) 61
B) 67
C) 72
D) 73
E) 77